# Lecture 4: More on Diamond

Lei Pan
School of Accounting, Economics and Finance
Curtin University

# Outline

- The Competitive Equilibrium (Solution of the Model)
  - Firms' profit maximisation
  - Individuals' utility maximisation
  - Market clearing conditions
  - Transition equation
  - Steady state and convergence to steady state
- Efficiency of the competitive equilibrium
  - Pareto-efficiency
  - Golden-rule
  - Capital under and over accumulation
  - Dynamic inefficiency

# The Competitive Equilibrium (Solution of the Model)

- Equations from last lecture:

$$L_t = (1+n)L_{t-1} \tag{1}$$

$$U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1}) \tag{2}$$

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta} \tag{3}$$

$$Y_t = F(K_t, A_t L_t^D), \quad y_t = f(k_t) \tag{4}$$

# Firm's Problem

- **Profit maximisation**

$$\max_{k_t, L_t^D} F(K_t, A_t L_t^D) - r_t K_t - w_t A_t L_t^D = A_t L_t^D (f(k_t) - r_t k_t - w_t)$$

FOCs:

$$r_t = f'(k_t) \tag{5}$$

$$w_t = f(k_t) - f'(k_t) k_t \tag{6}$$

# Individual's Problem (1 of 2)

- **Utility maximisation**

$$\max_{s_t} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1})$$

$$s.t \quad c_{1t} + s_t = w_t A_t \tag{7}$$

$$c_{2t+1} = (1 + r_{t+1}) s_t \tag{8}$$

- This problem can be transformed into a unconstrained maximisation problem:

$$\max_{s_t} U = u(w_t A_t - s_t) + \beta u[(1 + r_{t+1}) s_t]$$

Under our assumption on $u$, the objective function is strictly concave in $s_t$ (i.e. $\frac{\partial^2 U}{\partial s_t^2} < 0$), so the maximisation problem has a unique solution.

# Individual's Problem (2 of 2)

- The Euler equation:

$$\frac{U_1(c_{1t})}{U_2(c_{2t+1})} = \frac{u'(c_{1t})}{\beta u'(c_{2t+1})} = 1 + r_{t+1} \tag{9}$$

Combining the budget constraints (7)-(8) and the Euler equation (9), yields the equation that determines $s_t$:

$$\frac{u'(w_t A_t - s_t)}{\beta u'[(1 + r_{t+1})s_t]} = 1 + r_{t+1} \tag{10}$$

- The problem can also be formulated as a constrained maximisation problem:

$$\max_{s_t} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta u(c_{2t+1})$$

$$s.t. \quad c_{1t} + \frac{1}{1 + r_{t+1}} c_{2t+1} = A_t w_t \tag{11}$$

where Eq.(11) is an individual's lifetime budget constraint

# Restrictions on Utility Function

- An example to solve $s_t$: Let $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$ such that

$$u'(c) = c^{-\theta}$$

Therefore, Eq.(9) becomes:

$$\frac{c_{1t}^{-\theta}}{c_{2t+1}^{-\theta}} = \beta(1 + r_{t+1})$$

Substituting Eq.(7) and (8) into the equation above, can solve for $s_t$ (see Appendix 1):

$$s_t = \frac{\beta^{\frac{1}{\theta}}}{\beta^{\frac{1}{\theta}} + (1 + r_{t+1})^{1-\frac{1}{\theta}}} w_t A_t \tag{12}$$

- Note that the amount of saving is a fraction of the individual's total income ($w_t A_t$) when young. This fraction is the saving rate (endogenously determined!) denote as:

$$s(r_{t+1}) = \frac{\beta^{\frac{1}{\theta}}}{\beta^{\frac{1}{\theta}} + (1 + r_{t+1})^{1-\frac{1}{\theta}}} \in (0, 1) \tag{13}$$

# Saving Rate and Real Interest Rate

- How the saving rate $s(r)$ depends on $r$?
- First, find $s'(r)$ (see Appendix 2):

$$s'(r) = (\frac{1}{\theta} - 1)\frac{\beta^{\frac{1}{\theta}}(1+r)^{-\frac{1}{\theta}}}{[\beta^{\frac{1}{\theta}} + (1+r_{t+1})^{1-\frac{1}{\theta}}]^2}$$

- The sign of $s'(r)$ coincides with $\frac{1}{\theta} - 1$. Therefore,

$$s'(r) = \begin{cases} > 0 & \text{if } \theta < 1 \\ < 0 & \text{if } \theta > 1 \\ = 0 & \text{if } \theta = 1 \end{cases}$$

# Income Effect vs. Substitution Effect

- Notice that a rise in $r$ has both an income and substitution effect.
  - income effect: positive for $c_{1t}$ since consumer is a net saver
  - substitution effect: negative for $c_{1t}$ since the cost of the time $t$ consumption is higher
- When $\theta < 1$, the elasticity of substitution between consumption in the two periods $\frac{1}{\theta} > 1$, individuals are more willing to substitute consumption between two periods, hence substitution effect dominates, and individuals reduce their consumption when young and increase saving rate, so $s'(r) > 0$.
- When $\theta > 1$, the elasticity of substitution between consumption in the two periods $\frac{1}{\theta} < 1$, individuals are less willing to substitute consumption between two periods, hence income effect dominates, so $s'(r) < 0$.
- In the special case of $\theta = 1$, $s(r) = \frac{\beta}{1+\beta}$, $s'(r) = 0$, income and substitution effects exactly cancel out.

# Market Clearing Conditions (1 of 2)

- There are THREE markets: labour, capital, and output. Only need to consider market clearing conditions in two markets, then the market clearing condition for the third market would be automatically satisfied. (this is called Walras' Law, see Appendix 3 for proof)
- Total labour supplied by individuals in period $t$ is $L_t$ and total labour demanded by firms in period $t$ is $L_t^D$. Labour market clearing condition:

$$L_t^D = L_t \tag{14}$$

- Total savings carried to period $t+1$ by individuals is $L_t s_t$. Total capital demanded by firms in period $t+1$ is $K_{t+1}$. Capital market clearing condition:

$$K_{t+1} = L_t s_t \tag{15}$$

# Market Clearing Conditions (2 of 2)

- In the Solow-Swan Model, the equilibrium is characterised by a transition equation that describe how $k_t$ evolves over time. We use CRRA utility function to illustrate how we can derive it.
- For the CRRA utility, $s_t = s(r_{t+1})A_t w_t$, where $s(r_{t+1})$ is defined in Eq.(13). Hence, the market clearing condition for capital market is:

$$K_{t+1} = L_t s(r_{t+1}) A_t w_t \tag{16}$$

- Dividing both sides of Eq.(16) by $A_{t+1}L_{t+1}$ yields:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t \tag{17}$$

# Transition Equation

- Recall from Eq.(5) and (6):

$$r_{t+1} = f'(k_{t+1}), \quad w_t = f(k_t) - f'(k_t)k_t$$

Substituting $r_{t+1}$ and $w_t$ into Eq.(17), get:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s[f'(k_{t+1})][f(k_t) - f'(k_t)k_t]$$

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{\beta^{\frac{1}{\theta}}}{\beta^{\frac{1}{\theta}} + [1 + f'(k_{t+1})]^{1-\frac{1}{\theta}}} [f(k_t) - f'(k_t)k_t] \quad (18)$$

Notice that Eq.(18) is the transition equation, which implicitly defines $k_{t+1}$ as a function of $k_t$.

- With given $k_0 = \frac{K_0}{A_0 L_0}$ (where $L_0 = (1+n)L_{-1}$), and $A_t$ and $L_t$ evolving exogenously, Eq.(18) fully characterises the equilibrium dynamics of the system.

# Steady State

- Steady state and the convergence to steady state
  - If there exists a $k^*$ such that $k_{t+1} = k_t = k^*$ satisfying the transition equation (18), then such a $k^*$ is steady state value of $k_t$.
  - Once $k_t$ converges to $k^*$, the economy reaches its stationary equilibrium or balanced growth path.
  - Properties of the economy on its balanced growth path are similar as those of the Solow-Swan economy on its balanced growth path.

# Features of Balanced Growth Path

- The interest rate is a constant ($r_t = f'(k^*)$) such that the saving rate ($s(r_{t+1})$) is a constant (but endogenously determined).
- Wage rate per worker $A_t w_t$ grows at rate $g$.
- Aggregate output, capital stock, consumption and the investment all grow at rate $n + g$.
- Output per worker and capital per worker all grow at rate $g$. The capital output ratio is a constant.
- A young individual's saving grows at rate $g$ ($s_t = \frac{K_{t+1}}{L_t}$), such that a young individual's consumption ($c_{1t}$) and old individual's consumption ($c_{2t+1}$) both grow at rate $g$.

# Steady State of OLG Model

**Proposition**

In the OLG model with two-period lived households, Cobb-Douglas technology, and CRRA preferences, there exists a **unique** steady-state equilibrium with the capital-labour ratio $k^*$, and for any $\theta > 0$, this equilibrium is **globally stable** for all $k(0) > 0$.

# Proof of Proposition

- Now consider a special case of logarithmic utility ($\theta = 1$) and the Cobb-Douglas production ($f(k) = k^{\alpha}$). In this case,

$$s(r) = \frac{\beta}{1+\beta}$$
$$r_{t+1} = f'(k_{t+1}) = \alpha k_{t+1}^{\alpha-1}$$
$$w_t = f(k_t) - f'(k_t)k_t = k_t^{\alpha} - \alpha k_t^{\alpha-1} k_t = (1-\alpha)k_t^{\alpha}$$

- Therefore, the transition equation (18) becomes:

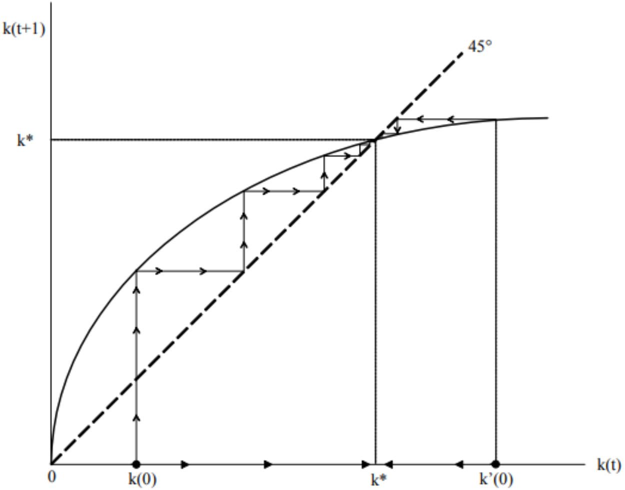
$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{\beta}{1+\beta} (1-\alpha) k_t^{\alpha} \tag{19}$$

- A unique non-zero steady state of $k$ is given by:

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(1+\frac{1}{\beta})}\right]^{\frac{1}{1-\alpha}} \tag{20}$$

- In this case, the convergence to $k^*$ is globally stable.
  - Wherever $k_t$ starts (other than 0), it converges to $k^*$.
  - The convergence is faster when $k_t$ is further away from $k^*$.

# Phase Diagram of the Canonical OLG Model



# Multiple Steady States

- The Solow-Swan economy has a unique balanced growth path and it is globally stable.
- However, with a general production function and CRRA utility function, it is possible that the Diamond economy has multiple steady states and some steady states are NOT stable.

# Pareto Optimality of the Competitive Equilibrium

- **Question**: Is the competitive equilibrium Pareto-efficient or Pareto optimal?
- Pareto efficiency
  - Definition: An allocation is Pareto-efficient if it is feasible and there is no other feasible allocation that increase one party's welfare without hurting another party's welfare. (Here, a feasible allocation means the allocation satisfies the resource constraint of the economy.)
  - In general, Pareto-efficient allocations are feasible allocations that maximise some special welfare function.
  - If an equilibrium allocation is not Pareto-efficient, there is space for government intervention to improve social welfare.

## $k^*$ vs. $k_{GR}$ (1 of 2)

The **golden-rule** allocation:

- To evaluate the Pareto-efficiency of the competitive equilibrium, we compare it with the golden-rule allocation. Specifically, we compare $k^*$ with the golden rule $k_{GR}$.
- Recall that $k_{GR}$ maximises steady state consumption, as in the Solow-Swan model, we consider the resource constraint of the economy, which states that the sources and uses of goods in an economy must be equal.
- In period $t$, the resource constraint for Diamond economy is:

$$F(K_t, A_t L_t) = L_t c_{1t} + L_{t-1} c_{2t} + (K_{t+1} - K_t) \tag{21}$$

where $K_{t+1} - K_t$ is the total investment in period $t$ ($I_t$).

## $k^*$ vs. $k_{GR}$ (2 of 2)

- For simplicity, let $A_t = A$ for all $t$ from now on, i.e. $g = 0$. Define $k_t = \frac{K_t}{AL_t}$, then dividing both sides of Eq.(21) by $AL_t$ gives:

$$f(k_t) = \frac{c_{1t}}{A} + \frac{c_{2t}}{A(1+n)} + [(1+n)k_{t+1} - k_t]$$

- In a steady state, $k_t$, $\frac{c_{1t}}{A}$, and $\frac{c_{2t}}{A}$ are all constant: $k_t = k$, $\frac{c_{1t}}{A} = \frac{c_1}{A}$, $\frac{c_{2t}}{A} = \frac{c_2}{A}$ for all $t$. Then the above resource constraint is reduced to:

$$\frac{c_1}{A} + \frac{c_2}{A(1+n)} = f(k) - nk \tag{22}$$

- Eq.(22) is the stationary resource constraint for the Diamond economy with $A_t = A$ for all $t$.

## Is $k_{GR}$ Pareto-efficient?

- It is clear from Eq.(22) that for $k_{GR}$ to maximise steady state consumption of individuals, it must maximise $f(k) - nk$. That is, $k_{GR}$ satisfies:

$$f'(k_{GR}) = n \tag{23}$$

- Is the golden-rule allocation Pareto-efficient?
- It can be shown that the golden rule allocation maximises the steady state lifetime utility of future generations among all stationary feasible allocations. That is, it solves the maximisation problem below (see Appendix 4).

$$\max_{c_1, c_2, k} U(c_1, c_2)$$

$$s.t \qquad c_1 + \frac{c_2}{1+n} = f(k) - nk$$

- This implies that the golden-rule allocation is Pareto-efficient.

# Is Competitive Equilibrium Pareto-efficient?

- We discuss this by comparing $k^*$ with $k_{GR}$. First, note that there is no guarantee that $k^* = k_{GR}$. For example, for logarithmic utility and Cobb-Douglas production:

$$k^* = [\frac{1-\alpha}{(1+n)(1+g)(1+\frac{1}{\beta})}]^{\frac{1}{1-\alpha}}$$

So,

$$f'(k^*) = \alpha {k^*}^{\alpha-1} = \frac{\alpha}{1-\alpha}(1+n)(1+g)(1+\frac{1}{\beta})$$

- It is clear that there is no guarantee that $f'(k^*) = n$. However, this does not necessarily mean the equilibrium is not Pareto-efficient. Let us discuss two possible cases: $k^* < k_{GR}$, $k^* > k_{GR}$.

# $k^* < k_{GR}$

- This is called under-accumulation of capital.
  - In this case, by concavity of $f(\cdot)$, we have $r_t = f'(k^*) > f'(k_{GR}) = n$, i.e. the interest rate is greater than the population growth rate.
  - How can the government encourage saving to increase capital?
  - The government can encourage saving by subsidising young individuals. To maintain a balanced budget, the government must tax old individuals to fund the subsidy. Would the current old generation like this plan?
  - Since the old attempt to encourage saving would hurt current old generation, a competitive equilibrium with under-accumulation of capital is Pareto-efficient.

# $k^* > k_{GR}$

- This is called over-accumulation of capital.
  - In this case, by concavity of $f(\cdot)$, we have $r_t = f'(k^*) < f'(k_{GR}) = n$, i.e. the interest rate is lower than the population growth rate.
  - The economy is said to be dynamically inefficient.
  - In this case, the competitive equilibrium is NOT Pareto-efficient.

# Proposition

**Proposition**

In the baseline OLG economy, the **competitive equilibrium is NOT necessarily Pareto optimal**. More specifically, when $r_t < n$, the economy is dynamically inefficient. In this case, it is optimal to reduce the capital stock starting from the competitive steady state and increase the consumption level of all generations.

# Why Inefficiency?

- Pecuniary externalities are important.
  - Individuals from generation $t$ face wages determined by the savings (capital stock) decisions of those from generation $t-1$.
  - An individual from generation $t-1$ receives a rate of return on her savings determined by the savings decisions of others of generation $t-1$.
- Dynamic inefficiency arises from overaccumulation that results from the need of the current generation to save for old age. The more they save, the lower is the return, the more they are encouraged to save.
- Notice that the possibility of inefficiency also stems from the dynamic population structure in the economy.
- Thus, if there was some way available to provide for the consumption when old, overaccumulation problem might be ameliorated.

# The Role of Social Security in Capital Accumulation

- Pay-as-you-go (unfunded) system: transfers from the young go directly to the current old. Discourages aggregate savings; in case of dynamic inefficiency, may lead to a Pareto improvement.
- Consider the "Pay-as-you-go-social security", if the time horizon for the economy is finite, $t = 0, 1, .., T$, i.e. generations born in period $T$ would die when young . Would the "Pay-as-you-go-social security" still work?
- It is clear that such a policy would hurt generation $T$ young individuals. So if the time horizon is finite, the equilibrium with capital over-accumulation would also be Pareto-efficient. Therefore, the possibility of inefficiency stems from the dynamic population structure in the Diamond model.

# Appendix 1 (1 of 2)

Let $u(c) = \frac{c^{1-\theta}}{1-\theta}$, show that $s_t = \frac{\beta^{\frac{1}{\theta}}}{\beta^{\frac{1}{\theta}} + (1+r_{t+1})^{1-\frac{1}{\theta}}} w_t A_t$.

**Solution**: The individual utility maximisation problem is formulated as:

$$\max_{c_{1t}, c_{2t+1}} U(c_{1t}, c_{2t+1}) = \frac{c_{1t}^{1-\theta} - 1}{1-\theta} + \beta \frac{c_{2t+1}^{1-\theta} - 1}{1-\theta}$$

$$\begin{aligned} s.t. \quad c_{1t} + s_t &= w_t A_t \\ c_{2t+1} &= (1 + r_{t+1}) s_t \end{aligned}$$

FOC:

$$\begin{aligned} \frac{\partial U}{\partial s_t} &= U_1(c_{1t}, c_{2t+1}) \frac{\partial c_{1t}}{\partial s_t} + U_2(c_{1t}, c_{2t+1}) \frac{\partial c_{2t+1}}{\partial s_t} \\ &= c_{1t}^{-\theta}(-1) + \beta c_{2t+1}^{-\theta}(1 + r_{t+1}) = 0 \end{aligned}$$

Thus, we get the consumption Euler equation:

$$\frac{c_{1t}^{-\theta}}{c_{2t+1}^{-\theta}} = \beta(1 + r_{t+1})$$

# Appendix 1 (2 of 2)

i.e.

$$(\frac{c_{2t+1}}{c_{1t}})^{\theta} = \beta(1 + r_{t+1}) \quad \text{or} \quad \frac{c_{2t+1}}{c_{1t}} = \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}}$$

Substituting the budget constraints when young and when old into the equation above, can get:

$$\frac{(1 + r_{t+1})s_t}{w_t A_t - s_t} = \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}}$$

$$(1 + r_{t+1})s_t = \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}} w_t A_t - \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}} s_t$$

$$[1 + r_{t+1} + \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}}]s_t = \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}} w_t A_t$$

$$s_t = \frac{\beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}}}{1 + r_{t+1} + \beta^{\frac{1}{\theta}}(1 + r_{t+1})^{\frac{1}{\theta}}} w_t A_t$$

$$s_t = \frac{\beta^{\frac{1}{\theta}}}{\beta^{\frac{1}{\theta}} + (1 + r_{t+1})^{1-\frac{1}{\theta}}} w_t A_t$$

# Appendix 2

Find $s'(r)$

**Solution**:

$$s'(r) = \beta^{\frac{1}{\theta}}[-\frac{(1-\frac{1}{\theta})(1+r)^{1-\frac{1}{\theta}-1}}{(\beta^{\frac{1}{\theta}}+(1+r_{t+1})^{1-\frac{1}{\theta}})^2}]$$

$$= \beta^{\frac{1}{\theta}}[-\frac{(1-\frac{1}{\theta})(1+r)^{-\frac{1}{\theta}}}{(\beta^{\frac{1}{\theta}}+(1+r_{t+1})^{1-\frac{1}{\theta}})^2}]$$

$$= (\frac{1}{\theta}-1)\frac{\beta^{\frac{1}{\theta}}(1+r)^{-\frac{1}{\theta}}}{[\beta^{\frac{1}{\theta}}+(1+r)^{1-\frac{1}{\theta}}]^2}$$

# Appendix 3

Prove Walras' Law (i.e. If a price vector $\vec{p}^* = (p_1^*, p_2^*)$ clears the market for $g_1$, then the market for $g_2$ must be cleared.)

**Proof.**

First notice that Walras' Law guarantees that the value of aggregate excess demand is equal to zero (i.e. $D \equiv 0$) given any price. The mathematical expression is as below:

$$p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) = 0 \tag{A.1}$$

where $z(p)$ is the aggregate excess demand function

Plug $p_1^*$ and $p_2^*$ into Eq.(A.1), get:

$$p_1^* z_1(p_1^*, p_2^*) + p_2^* z_2(p_1^*, p_2^*) = 0 \quad \text{(Walras' Law holds for } (p_1^*, p_2^*))$$

But $(p_1^*, p_2^*)$ clears the market for $g_1$ (expressed as the equation below):

$$z_1(p_1^*, p_2^*) = 0 \tag{A.2}$$

Plug Eq.(A.2) into Eq.(A.1), we obtain:

$$0 + p_2^* z_2(p_1^*, p_2^*) = 0$$

Since $p_2^* > 0$, $z_2(p_1^*, p_2^*) = 0$. □

# Appendix 4 (1 of 2)

Show that the golden-rule allocation solves the following maximisation problem.

$$\max_{c_1,c_2,k} U(c_1,c_2)$$
$$s.t \qquad c_1 + \frac{c_2}{1+n} = f(k) - nk$$

**Solution**: Set up the Lagrangian function:

$$\mathcal{L} = U(c_1,c_2) + \lambda(f(k) - nk - c_1 - \frac{c_2}{1+n})$$

FOCs:

$$\frac{\partial \mathcal{L}}{c_1} = U_1(c_1,c_2) - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{c_2} = U_2(c_1,c_2) - \frac{\lambda}{1+n} = 0$$
$$\frac{\partial \mathcal{L}}{k} = \lambda(f'(k) - n) = 0$$

# Appendix 4 (2 of 2)

- From the FOC w.r.t. $c_1$, $\lambda = U_1(c_1, c_2) > 0$. Therefore, the FOC w.r.t. $k$ implies:

$$f'(k) = n$$

- This is exactly the condition that determines $k_{GR}$. With $k$ determined, $c_1$ and $c_2$ are determined by:

$$\frac{U_1(c_1, c_2)}{U_2(c_1, c_2)} = 1 + n$$

$$c_1 + \frac{c_2}{1 + n} = f(k) - nk$$

- In fact, this is the way I was taught in my honours year how to find the golden rule allocation. But for this unit, you are only required to know how to find $k_{GR}$ using the resources constraint of the economy.